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ON MATROID INTERSECTIONS

by

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This paper exploits and extends results of Edmonds, Cunningham, Cruse and McDiarmid on matroid intersections. Let r_1 and r_2 be rank functions of two matroids defined on the same set E. For every $S\subset E$, let $r_{12}(S)$ be the largest cardinality of a subset of S independent in both matroids, $0\leqq k\leqq r_{12}(E)-1$. It is shown that, if c is nonnegative and integral, there is a y: $2^E\to Z^+$ which maximizes $\sum_S (k-r_{12}(E-S))y(S)$ and $\sum_S (k+1-r_{12}(E-S))y(S)$, subject to $y\geqq 0$, $\forall j\in E, \sum_{S\ni j} y(S)\leqq c_j$.

1. Introduction

Let E be a finite set admitting two matroids, M_1 and M_2, with respective rank functions r_1 and r_2. For every $S\subset E$, let $r_{12}(S)$ denote the maximum cardinality of a subset $T\subset S$ which is independent in both matroids. We shall be considering, for every integer k, $0\leqq k\leqq r_{12}(E)$, the function $f_k(S)=k-r_{12}(E-S)$ and the family $\mathcal{I}_k$ of all subsets of k elements of E independent in both matroids. The following results, due to Edmonds ([3], [4]), and to Cunningham and McDiarmid ([2] and [10], see also [1]) respectively, deal with these concepts and motivate our further observations.

Theorem 1.1. *For every $S\subset E$,*

$$r_{12}(S) = \min_{T\subset S} r_1(T)+r_2(S-T). \tag{1.1}$$

Theorem 1.2. *Let E have n elements, labeled, $1, \dots, n$. Then the polyhedron of all $x=(x_1, \dots, x_n)$ satisfying*

$$x_j \geqq 0, \quad j=1, \dots, n$$

$$\forall S\subset E, \ \sum_{j\in S} x_j \geqq f_k(S)$$

has for its vertices the set of incidence vectors of sets in $\mathcal{I}_k$ (i.e., the vertices are precisely all (0, 1) vectors x with exactly k coordinates equal to 1 such that $\{j\,|\,x_j=1\}$ is independent in M_1 and M_2).

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To state our results, we first define, for any real valued function f defined on all subsets of E, $f^*(S)=\max \sum_i f(T_i)$, where $\{T_i\}$ range over all possible partitions of S. We shall prove the following theorems:

Theorem 1.3. *For every* $S\subset E$,

$$\min_{I\in \mathcal{I}_k} |S\cap I| = \max(0, f_k^*(S)).$$

Theorem 1.4. *For every* $S\subset E$ *and every integer* k, $0\leqq k\leqq r_{12}(E)-1$, *if* $f_k^*(S)$ *and* $f_{k+1}^*(S)$ *are not both nonpositive, there exists a partition* $\{T_1, \ldots, T_s\}$ *of* S *such that*

$$f_k^*(S) = \sum_i f_k(T_i) \quad \textit{and} \quad f_{k+1}^*(S) = \sum_i f_{k+1}(T_i).$$

Theorem 1.3 suggests that linear programming duality is applied to the polyhedron discussed in theorem 1.2. This is indeed the case, and we make use of the concept of lattice polyhedron [7] (together with theorems 1.1 and 1.2) by extending an idea described in [9]. (It is not surprising that the concept of lattice polyhedron is relevant here, since theorems 1.1 and 1.2 are corollaries of that concept).

Theorem 1.4 is an example of the "t-phenomenon" (transition phenomenon) noted in [5], [6] and [8]. Our proof of theorem 1.4 depends on exhibiting the t-phenomenon for lattice polyhedra.

2. Proof of Theorem 1.3.

Let L be a lattice, and let A be a (0, 1) matrix with rows A_a indexed by the elements of the lattice. Assume

(2.1) $$a < b < c, \quad A_{aj} = A_{cj} = 1 \quad \text{implies} \quad A_{bj} = 1; \quad \text{and}$$

(2.2) $$A_{a\vee b}+A_{a\wedge b} \leqq A_a+A_b \quad \text{for all} \quad a, b\in L.$$

Let $g\colon L\to R$ satisfy

(2.3) $$g(a\wedge b)+g(a\vee b) \geqq g(a)+g(b).$$

Consider the linear programming problem

(2.4) $$\max(g, y)|y'A \leqq c', \; y \geqq 0,$$

where $c\geqq 0$, integral.

Lemma 2.1 [7]. *If* (2.4) *has a solution, it has a solution* $\bar{y}$ *which is integral.*

Our strategy in proving Theorem 1.3 is to construct suitable L, A, g and c. First, we define L. Its elements are all ordered pairs (S_1, S_2), where $S_1, S_2\subset E$. We say

(2.5) $$(S_1, S_2) \leqq (T_1, T_2) \quad \text{if} \quad S_1 \subset T_1, \quad T_2 \subset S_2.$$

The partial ordering (2.5) clearly defines a lattice, with

$$(S_1, S_2)\vee(T_1, T_2) = (S_1\cup T_1, S_2\cap T_2),$$

$$(S_1, S_2)\wedge(T_1, T_2) = (S_1\cap T_1, S_2\cup T_2).$$

The columns of A are indexed by elements of E, and define

(2.6) $$A_{(S_1, S_2)j} = \begin{cases} 1 & \text{if } j \in S_1 \cap S_2 \\ 0 & \text{if } j \notin S_1 \cap S_2. \end{cases}$$

From (2.5) and (2.6), we infer (2.1) and (2.2). Now we define

(2.7) $$g(S_1, S_2) = k - r_1(E-S_1) - r_2(E-S_2),$$

and we now try to prove (2.3). Since r_1 and r_2 are rank functions of a matroid, we have

(2.8) $$r_i(S_1 \cup S_2) + r_i(S_1 \cap S_2) \leqq r_i(S_1) + r_i(S_2) \quad \text{for any} \quad S_1,\ S_2 \text{ and } \quad i = 1, 2.$$

Define $\bar{r}_i(S) = r_i(E-S)$.
From (2.8), it follows that

(2.9) $$\begin{aligned} \bar{r}_i(S_1 \cup S_2) + \bar{r}_i(S_1 \cap S_2) &= r_i(E-(S_1 \cup S_2)) + r_i(E-(S_1 \cap S_2)) \\ &= r_i((E-S_1) \cap (E-S_2)) + r_i((E-S_1) \cup (E-S_2)) \\ &\leqq r_i(E-S_1) + r_i(E-S_2) \\ &= \bar{r}_i(S_1) + \bar{r}_i(S_2), \quad i = 1, 2. \end{aligned}$$

Then (2.9) and (2.7) imply (2.3).

Next, let $T \subset E$. There are many choices of $S_1,\ S_2 \subset E$ such that $T = S_1 \cap S_2$. Let us observe that

(2.10) $$\max g(S_1, S_2) | S_1 \cap S_2 = T \quad \text{is} \quad k - r_{12}(E-T).$$

To prove (2.10), we must show

$$\min r_1(E-S_1) + r_2(E-S_2) | S_1 \cap S_2 = T \quad \text{is} \quad r_{12}(E-T).$$

By the monotonicity of each r_i, it is sufficient to show

(2.11) $$\min r_1(E-S_1) + r_2(E-S_2) | S_1 \cap S_2 = T, \quad S_1 \cup S_2 = E \quad \text{is} \quad r_{12}(E-T).$$

By theorem 1.1,

$$\begin{aligned} r_{12}(E-T) &= \min_{U \subset E-T} r_1(U) + r_2(E-T-U) \\ &= \min_{T \subset E-U} r_1(E-(E-U)) + r_2(E-(T \cup U)). \end{aligned}$$

Now, if $S_1 = E-U$, $S_2 = (T \cup U)$, then $S_1 \cap S_2 = T$, $S_1 \cup S_2 = E$. On the other hand, if $S_1 \cap S_2 = T$, $S_1 \cup S_2 = E$, let $U = E - S_1$. Then $S_1 = E-U$ contains T, and $T \cup U = S_2$. This proves (2.11), hence (2.10).

Now let $S \subset E$ be given and consider the linear programming problem:

(2.12) $$\max \sum_{T \subset E} f_k(T) y(T) | y(T) \geqq 0 \quad \text{for all } T;$$

$$\forall j \in S, \ \sum_{T \ni j} y(T) \leqq 1; \quad \forall j \in E-S, \ \sum_{T \ni j} y(T) \leqq 0.$$

By (2.10) and Lemma 2.1, this maximum is attained by a (0, 1) vector y such that, if $K=\{i\,|\,y_i=1\}$ is not empty, the sets $\{T_i\}_{i\in K}$ are disjoint subsets of S and the value of the maximum is $f_k^*(S)\geqq 0$. (Since f_k is monotonic, there is no restriction in assuming $\{T_i\}$ partition S.) If K is empty, the maximum is 0. On the other hand, the dual program, by Theorem 1.2, has value $\min_{I\in\mathcal{I}_k}|S\cap I|$. Thus theorem 1.3 follows from the duality theorem.

3. Proof of Theorem 1.4.

Assume a lattice L, a (0, 1) matrix A, and a function $g\colon L\to\mathbf{Z}$ satisfying (2.1)—(2.3). Assume $c\geqq 0$ and integral. Note that, if g satisfies (2.3), the function $g+\vec{1}$ does also.

Theorem 3.1. *Assume that the polyhedron*

$$\{x|Ax \geqq g+\vec{1},\ x \geqq 0\} \quad \textit{is not empty.} \tag{3.1}$$

Then there exists an integral vector $\bar{y}$ which is an optimum solution to both linear programming problems:

$$\begin{aligned} &\max\,(g, y)|y \geqq 0,\ y'A \leqq c', \quad \textit{and} \\ &\max\,(g+\vec{1}, y)|y \geqq 0,\ y'A \leqq c'. \end{aligned} \tag{3.2}$$

It should be clear from the arguments used in section 2 that Theorem 3.1 includes Theorem 1.4, so we prove theorem 3.1.

Let B be the matrix A with an additional column of 1's appended, and consider the polyhedron

$$Q \equiv \{\hat{x}|B\hat{x} \geqq g+\vec{1},\ \hat{x} \geqq 0\}. \tag{3.3}$$

Next, let w_0 and w_1 be the values of the respective two linear programming problems mentioned in (3.2). Then $w_1\geqq w_0$ and let $\Delta=w_1-w_0$.

Observe that every $\hat{x}$ in Q can be written $(x; v)$, where v is a scalar, and consider the linear programming problem:

$$\min\,(c, x)+\Delta v|\hat{x} = (x; v)\in Q. \tag{3.4}$$

By the theory of lattice polyhedra, every vertex of Q is integral, so (3.4) has a solution where v is a nonnegative integer. We shall show that there is a solution $\hat{x}$ for which $v=1$. To do this, let

$$m(v) = \min\,(c, x)|Ax+v\vec{1} \geqq g+\vec{1},\ x \geqq 0,\ v \geqq 0.$$

Clearly, $m(0)+\Delta\cdot 0=w_1$, $m(1)+\Delta\cdot 1=w_0+\Delta=w_1$. So all we need show is that

$$m(v) \geqq w_1-\Delta v \quad \text{for all } v > 1. \tag{3.5}$$

By the definition of $m(v)$, there is a vector x^v such that

$$Ax^v \geqq g+(1-v)\vec{1},\ x^v \geqq 0$$

$$(c, x^v) = m(v).$$

By the definition of w_1, there is an x^1 such that

$$Ax^1 \geqq g+\vec{1},\ x^1 \geqq 0$$

$$(c, x^1) = w_1.$$

Let $x=\frac{1}{v}x^v+\frac{v-1}{v}x^1$. Then

$$Ax \geqq g,\ x \geqq 0 \tag{3.6}$$

$$(c, x) = \frac{1}{v}\big(m(v)+(v-1)w_1\big).$$

Comparison of (3.6) with (3.2) shows

$$w_0 \leqq \frac{1}{v}\big(m(v)+(v-1)w_1\big)$$

which is (3.5). Therefore, there is a solution of (3.4) with value w_1, in which $v=1$.

The dual of (3.4) is

$$\max(g+\vec{1}, y) | y \geqq 0,\ y'A \leqq c',\ (\vec{1}, y) \leqq \Delta. \tag{3.7}$$

By complementary slackness, since $v=1$ the integral vector y solving (3.7) has $(\vec{1}, y)=\Delta$. We know $(g+\vec{1}, y)=w_1$, therefore $(g, y)=w_0$, and we are done

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